

# MATH 590 REPORT ON WARMSTARTING INTERIOR-POINT METHODS FOR LINEAR PROGRAMMING

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ABSTRACT. In this report, we give a summary of the paper *A Primal-Dual Slack Approach to Warmstarting Interior-Point Methods for Linear Programming*. We start this report with a short summary about Interior-Point Methods. We then introduce the concept of *warmstarting* a Linear Program. We explain the two main problems when warmstarting with an IPM. We then present the slack approach proposed by Engau *et. al.* to efficiently warmstart with an IPM. The last part of this report treats the choice of slack variables for the proposed method. We conclude with some ideas for future research.

## 1. INTRODUCTION

A linear program (LP) in standard primal form is

$$(1) \quad \begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

where  $x \in \mathbb{R}^n$  is the decision variable,  $A \in \mathbb{R}^{m \times n}$  is the constraint matrix,  $b \in \mathbb{R}^m$  is the right-hand-side vector, and  $c \in \mathbb{R}^n$  is the cost vector. We denote  $d = (A, b, c)$  as the problem data for (1). The dual problem of (1) has the form

$$(2) \quad \begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & A^\top y + s = c \\ & s \geq 0, \end{aligned}$$

where  $y \in \mathbb{R}^m$  is the dual variable, and  $s \in \mathbb{R}^n$  is the slack variable.

In this report, we look at primal-dual path-following Interior-Point Methods (IPMs), as explained in [2, 7], that are based on the logarithmic barrier function

$$(3) \quad \min c^\top x - \mu \sum_{i=1}^n \log(x_i) \quad \text{s.t.} \quad Ax = b.$$

The first-order Karush-Kuhn-Tucker (KKT) optimality conditions for the barrier function are

$$(4a) \quad Ax = b$$
$$(4b) \quad A^\top y + s = c$$
$$(4c) \quad XSe = \mu e,$$

where  $e = (1, 1, \dots, 1)^\top$ . Here, and henceforth variables represented by capital letters will represent the diagonal matrix generated by the corresponding lower-case variable, for example  $X = \text{diag}(x)$ ,  $S = \text{diag}(s)$ . Equation (4a) ensures the primal feasibility, (4b) ensures the dual feasibility, and (4c) represents the complementary slackness. In [8] it is shown that

for every  $\mu > 0$ , there is a unique solution to (4). The so-called central path is the set of  $x(\mu), \mu > 0$  and converges to the optimal solution for (1) when  $\mu$  approaches zero. Under condition (4) we have that

$$n\mu = x^\top s = x^\top (c - A^\top y) = x^\top c - x^\top A^\top y = c^\top x - b^\top y,$$

so it follows that  $n\mu$  represents the duality gap between the primal and the dual solution for the penalized problem.

In the primal-dual algorithm, to save computation, we solve the KKT conditions (4) only approximately. We use Newton's method to compute a step  $\Delta z = (\Delta x, \Delta y, \Delta s)$ . However, for (4c), we approximate the non-linear equation

$$(x_i + \Delta x_i)(s_i + \Delta s_i) = s_i \Delta x_i + x_i \Delta s_i + \Delta x_i \Delta s_i + x_i s_i = \mu$$

by dropping the non-linear term with

$$s_i \Delta x_i + x_i \Delta s_i = \mu - x_i s_i.$$

We can therefore solve the following Newton system

$$(5) \quad \begin{bmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} r_b \\ r_c \\ \mu e - X S e \end{bmatrix}$$

where  $r_b = b - Ax$  and  $r_c = c - A^\top y - s$  are the residuals. They are zero if the starting point is feasible, non-zero if the starting point is infeasible. To solve this system we use the equation in the second row to obtain  $\Delta s = r_c - A^\top \Delta y$  and substitute in the equation of the last row to obtain

$$S \Delta x - X A^\top \Delta y = \mu e - X S e - X r_c.$$

Multiplying this equation by  $AS^{-1}$  and using  $A \Delta x = r_b$ , we get

$$AS^{-1} X A^\top \Delta y = r_b - AS^{-1}(\mu e - X S e - X r_c).$$

Let the diagonal matrix  $D = S^{-1/2} X^{-1/2}$ . Then the solution, the so-called normal equations, becomes

$$(6) \quad \begin{aligned} \Delta y &= (AD^2 A^\top)^{-1} (r_b - AS^{-1}(\mu e - X S e - X r_c)) \\ \Delta s &= r_c - A^\top \Delta y \\ \Delta x &= S^{-1}(\mu e - X S e) - D^2 \Delta s. \end{aligned}$$

Commonly, algorithms either solve (6) by using a Cholesky factorization of  $AD^2 A^\top$ , or they solve the symmetric indefinite system

$$(7) \quad \begin{bmatrix} 0 & A \\ A^\top & -D^{-2} \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \end{bmatrix} = \begin{bmatrix} r_b \\ r_c - \mu X^{-1} e + s \end{bmatrix}.$$

Since the solution to (5) is only an approximation to the KKT conditions, we perform a line-search in the direction of the Newton step. The details for a line-search algorithm can be found in [2]. Following is an outline of the primal-dual path-following algorithm.

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**Algorithm 1:** Primal-Dual Path-Following IPM

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**Input:**  $d = (A, b, c)$ ,  $z^0 = (x^0, y^0, s^0)$ ,  $\mu > 0$ ,  $\rho < 1$ ,  $\epsilon_b > 0$ ,  $\epsilon_c > 0$ ,  $\epsilon_d > 0$

**Output:** Optimal solution  $z^*$

**begin**

$k \leftarrow 0$ ;

$r_b^k = b - Ax^k$ ;

$r_c^k = c - A^\top y^k - s^k$ ;

**while**  $\max \left\{ \frac{\|r_b^k\|}{\epsilon_b}, \frac{\|r_c^k\|}{\epsilon_c}, \frac{(x^k)^\top s^k}{\epsilon_d} \right\} > 1$  **do**

        Solve the Newton system (5) for  $\Delta z = (\Delta x, \Delta y, \Delta s)$ ;

        Perform linesearch to compute step size  $\alpha^k$ ;

$z^{k+1} = (x^{k+1}, y^{k+1}, s^{k+1}) \leftarrow (x^k, y^k, s^k) + \alpha^k(\Delta x, \Delta y, \Delta s)$ ;

$k \leftarrow k + 1$ ;

        Update  $r_b^k, r_c^k$ ;

$\mu^k \leftarrow \rho \mu^{k-1}$

**end**

$z^* \leftarrow z^k$ ;

**end**

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At each iteration in algorithm 1, we compute a step length  $\alpha$  for the Newton step. Now given that  $r_b^k = b - Ax^k$ ,  $A\Delta x^k = r_b^k$ , and  $x^{k+1} = x^k + \alpha\Delta x^k$ , we have

$$\begin{aligned} r_b^{k+1} &= b - Ax^{k+1} \\ &= b - A(x^k + \alpha\Delta x^k) \\ &= b - Ax^k - \alpha A\Delta x^k \\ &= b - Ax^k - \alpha r_b^k \\ &= r_b^k - \alpha r_b^k \\ &= (1 - \alpha)r_b^k. \end{aligned}$$

We therefore have

$$(8) \quad r_b^{k+1} = (1 - \alpha^k)r_b^k = \prod_{j=0}^k (1 - \alpha^j)r_b^0.$$

Hence a feasible IPM will always maintain feasibility of its starting points, and an infeasible IPM (IIPM) can only establish feasibility by taking a full Newton step with  $\alpha^k = 1$ .

## 2. WARMSTARTING INTERIOR-POINT ALGORITHMS

In many real-life applications where an instance of problem (1) needs to be solved, it often occurs that we already have an estimate or an approximation for the solution that we would like to exploit in order to accelerate the solving time. A classic example is *portfolio optimization* where one optimizes the statistical *expectation* of a portfolio, subject to constraints that involve the *covariance* of the portfolio data [5]. The portfolio data consists usually of daily stock market information about the individual stocks over a period of time. Depending on the size of the portfolio, solving (1) can take a long time. At the end of a day, the investor would like to incorporate the new market information of the current day into the model, in order to solve for a new optimal investment for the next day. If the historical data in the model is large, such changes result in small perturbations of the original problem.

Let us consider an initial LP with data  $d^\circ = (A^\circ, b^\circ, c^\circ)$  with a known optimal solution  $z^\circ = (x^\circ, y^\circ, z^\circ)$ . Let  $\Delta d = (\Delta A, \Delta b, \Delta c)$  be the perturbation of the original problem  $d^\circ$ . Our new problem is therefore  $d = (A, b, c) = (A^\circ + \Delta A, b^\circ + \Delta b, c^\circ + \Delta c)$ . Warmstarting means to use the solution  $z^\circ$  as a starting point for a new problem with data set  $d$ . There are two main issues when warmstarting an IPM for an LP.

- (a) Since  $z^\circ$  is optimal for  $d^\circ$ , it satisfies the KKT conditions and it follows that  $r_b = r_c = 0$ , and therefore

$$r_b = b - Ax^\circ = (b^\circ + \Delta b) - (A^\circ + \Delta A)x^\circ = \Delta b - \Delta Ax^\circ$$

and

$$r_c = c - A^\top y^\circ - s = (c^\circ + \Delta c) - (A^\circ + \Delta A)^\top y^\circ - s^\circ = \Delta c - \Delta A^\top y^\circ$$

where  $r_b$  and  $r_c$  are now potentially nonzero. Hence our starting point  $z^\circ = (x^\circ, y^\circ, s^\circ)$  may not satisfy (4a) and (4b).

- (b) Since  $z^\circ$  is an optimal point, we have  $x^\circ s^\circ \approx 0$ . Hence if  $\mu^0 > 0$  then  $X^0 S^0 e < \mu^0 e$  and the point  $z^\circ$  violates the interiority condition  $(x^0, s^0) > 0$ . The strict feasibility  $(x^0, s^0) > 0$  is required to apply IPMs (hence the name *Interior* Point Methods).

One way to address (a) is to project onto the feasible set. This was proposed in [6] by solving the weighted least-squares adjustments (LSA)

$$(9) \quad \begin{aligned} & \min_{\Delta x} \|\Sigma \Delta x\| \\ & \text{s.t. } A(x^\circ + \Delta x) = b \\ & \quad x^\circ + \Delta x \geq 0 \end{aligned}$$

and

$$(10) \quad \begin{aligned} & \min_{\Delta y, \Delta s} \|\Lambda \Delta s\| \\ & \text{s.t. } A^\top (y^\circ + \Delta y) + (s^\circ + \Delta s) = c \\ & \quad s^\circ + \Delta s \geq 0, \end{aligned}$$

where  $\Sigma$  and  $\Lambda$  are chosen according to one of the following methods.

- *Plain LSA* (PLSA):  $\Sigma$  and  $\Lambda$  are identity matrices.
- *Weighted LSA* (WLSA):  $\Sigma = X^{-1}$  and  $\Lambda = S^{-1}$ .
- *Jointly WLSA* (JWLSA):  $\Sigma = X^{-1}S$  and  $\Lambda = S^{-1}X$ .

The new point  $(x, y, s) = (x^\circ + \Delta x, y^\circ + \Delta y, s^\circ + \Delta s)$  obtained by the projection will be feasible. However, the point will not be in the interior, in general and the method does therefore not solve (b). Nevertheless, the method works well for problem instances, where the projected point might be close enough to the optimal solution and there is no need to use an IPM to converge.

A second way to address (a) is to solve directly the Newton system

$$\begin{aligned} A\Delta x &= r_b \\ A^\top \Delta y + \Delta s &= r_c \\ X^\circ \Delta s + S^\circ \Delta x &= 0. \end{aligned}$$

To solve this system, it needs to be reduced to the equivalent normal equations. It can then be solved with the Cholesky factorization of  $AD^2A^\top$ . Similar to the projection method, this

approach will result in a feasible point, but it will not result in a sufficiently interior point to apply an IPM.

An approach to correct (a) and (b) simultaneously was proposed in [1]. Auxiliary variables  $\xi$  and  $\psi$  are introduced to relax the non-negativity constraint on  $x$  and  $s$ . Additional penalty parameters  $t$  and  $u$  penalize the auxiliary variables in the objective function. The original problem becomes

$$\begin{array}{ll} \min c^\top x + t^\top \xi & \max b^\top y - u^\top \psi \\ \text{s.t. } Ax = b & \text{s.t. } A^\top y + s = c \\ 0 \leq x + \xi \leq u & -\psi \leq s \leq t \\ \xi \geq 0 & \psi \geq 0. \end{array}$$

If the penalty parameter  $t$  and  $u$  are chosen sufficiently large, then the IPM will eventually reduce the auxiliary variables to zero. However, the correct choice of the penalty parameters is the drawback of this method. Too high values for  $t$  and  $u$  result in numerical instability. Too low values might leave the auxiliary variables non-zero and the solution is not optimal.

### 3. WARMSTART APPROACH BY ENGAU *et. al.*

In [4], Engau, Anjos, and Vannelli introduce a new set of slack variables  $\xi$  and  $\psi$ . The resulting warmstarting scheme becomes

$$(11) \quad \begin{array}{ll} \min c^\top x & \max b^\top y \\ \text{s.t. } Ax = b & \text{s.t. } A^\top y + s = c \\ x - \xi = 0 & s - \psi = 0 \\ \xi \geq 0 & \psi \geq 0. \end{array}$$

The advantage of this formulation is that there are no penalty parameters needed. Note also that  $x$  and  $s$  are unrestricted and  $X^\circ S^\circ \approx 0$  does not affect the interiority anymore. Hence we can set  $z^0 = z^\circ$  as long as we choose  $\xi^0$  and  $\psi^0$  in a way that they are sufficiently interior. The new KKT conditions for (11) are therefore

$$\begin{array}{ll} (12a) & Ax = b \\ (12b) & A^\top y + s = c \\ (12c) & x - \xi = 0 \\ (12d) & s - \psi = 0 \\ (12e) & \Xi \Psi e = \mu e, \end{array}$$

where  $\Xi \Psi e = \mu e$  now measures our duality gap. Conditions (12c) and (12d) ensure that our final feasible solution, when  $\mu = 0$ , will be the same as the solution to (4). In fact, solving for the Newton step in our first iteration, should already reduce the residuals  $r_x = \xi - x$  and  $r_s = \psi - s$  close to zero so that  $x^1 \approx \xi^1$  and  $s^1 \approx \psi^1$ . Using again the same residuals  $r_b$  and  $r_c$  as in (5), we can rewrite our Newton system as

$$(13) \quad \begin{bmatrix} A & 0 & 0 & 0 & 0 \\ 0 & A^\top & I & 0 & 0 \\ I & 0 & 0 & -I & 0 \\ 0 & 0 & I & 0 & -I \\ 0 & 0 & 0 & \Psi & \Xi \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \\ \Delta \xi \\ \Delta \psi \end{bmatrix} = \begin{bmatrix} r_b \\ r_c \\ r_x \\ r_s \\ \mu e - \Xi \Psi e \end{bmatrix}.$$

If we let  $\Delta\xi = \Delta x - r_x$ ,  $\Delta\psi = \Delta s - r_s$ , and  $R_x = \text{diag}(r_x)$ ,  $R_s = \text{diag}(r_s)$ , we can reduce (13) to

$$(14) \quad \begin{bmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ \Psi & 0 & \Xi \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} r_b \\ r_c \\ \mu e - \Xi\Psi e + \Xi R_s e + \Psi R_x e \end{bmatrix}$$

which has the same structure as (5). If, this time, we let  $D = \Xi^{1/2}\Psi^{-1/2}$ , we can further reduce the system to the normal equations

$$(15) \quad \begin{aligned} \Delta x &= A^{-1}r_b \\ \Delta y &= (A^{-1})^\top(r_c + \mu\Xi^{-1}e - \Psi + r_s + \Xi^{-1}\Psi r_x) + (A^{-1})^\top D^{-2}\Delta x \\ \Delta s &= r_c - A^\top \Delta y \\ \Delta\xi &= \Delta x - r_x \\ \Delta\psi &= \Delta s - r_s. \end{aligned}$$

The computation for the first three equations is about the same as for (6) in the original problem. The additional computation of  $\Delta\xi$  and  $\Delta\psi$  is negligible. Hence, the new system will not increase the computation time for further iterates, once the points are well centered again.

In their paper, Engau *et. al.* show that for solving (1), the overall complexity of an IPM is maintained or even improved [4]. They also show that the warmstart residuals  $\|r_b^0\|$  and  $\|r_c^0\|$  are smaller than for a cold start and that the algorithm needs therefore fewer iterations to converge than a cold start, if the perturbations are not too big.

#### 4. INITIALIZING THE SLACK VARIABLES

In the complexity analysis in [4], it is assumed that  $(\xi^0, \psi^0) = (\zeta e, \zeta e)$ , where  $\zeta \geq \|(\xi^*, \psi^*)\|_\infty = \|(x^*, s^*)\|_\infty$  and  $(x^*, s^*)$  is an optimal solution. It is clear that  $(x^*, s^*)$  is not known in advance, and hence, the initial point for  $(\xi^0, \psi^0)$  must be chosen differently. There are two main factors in choosing the initial point.

- Since  $X^\circ S^\circ \approx 0$ , we do not want to carry the numerical difficulties of very small entries in either  $x^\circ$  or  $s^\circ$ . We therefore require

$$(\xi^0, \psi^0) \geq (x^\circ, s^\circ).$$

- From (8) we know that  $r^{k+1} = (1 - \alpha)r^k$  for all the residuals. It is therefore desirable that the residuals  $r_x^0 = \xi^0 - x^\circ$  and  $r_s^0 = \psi^0 - s^\circ$  start at a similar size as  $r_b^0$  and  $r_c^0$ . Engau *et. al.* therefore suggest

$$(16) \quad \begin{aligned} r_x^0 &= (X^\circ)^{1/2}(S^\circ)^{-1/2}\|(r_b, r_c, 1)\|_\infty e \\ r_s^0 &= (X^\circ)^{-1/2}(S^\circ)^{1/2}\|(r_b, r_c, 1)\|_\infty e, \end{aligned}$$

where  $\|(r_b, r_c, 1)\|_\infty$  measures the maximal residual.

We can therefore initialize  $\xi^0$  and  $\psi^0$  with

$$\begin{aligned}
\Xi^0 \Psi^0 e &= (X^\circ + R_x^0)(S^\circ + R_s^0)e \\
&= X^\circ S^\circ + R_x^\circ S^\circ + X^\circ R_s^\circ + R_x^\circ R_s^\circ \\
&= \mu e + 2(X^\circ)^{1/2}(S^\circ)^{-1/2} S^\circ \|(r_b, r_c, 1)\|_\infty e + \|(r_b, r_c, 1)\|_\infty^2 e \\
&= \mu e + 2\sqrt{\mu} \|(r_b, r_c, 1)\|_\infty e + \|(r_b, r_c, 1)\|_\infty^2 e \\
&= (\sqrt{\mu} + \|(r_b, r_c, 1)\|_\infty)^2 e.
\end{aligned}$$

Note that this warmstart method could not be used from an exact optimal solution  $(x^\circ, s^\circ)$  since zero components in  $x^\circ$  or  $s^\circ$  would render the inverses of  $X^\circ$  and  $S^\circ$  infeasible. Also, an approximate optimal solution with a small centrality measure  $\mu = x^{\circ\top} s^\circ \gtrsim 0$  will result in a large starting point that slows down the convergence of the algorithm.

We can avoid such problems by not relaxing variables that do not need to be relaxed. In an exact solution, we call the variables that are zero non-basic variables. Non-basic variables need not to be relaxed. So if we could determine the variables that are non-basic, we could directly warmstart from their previous solution. Among many different indicator functions for basic variables that are found in the literature, Engau *et. al.* suggest the primal-dual indicator defined by

$$(17) \quad \frac{x_i^\circ}{s_i^\circ} < \tau \Rightarrow \quad x_i^\circ \rightarrow 0 \text{ and } s_i^\circ > 0$$

$$(18) \quad \frac{s_i^\circ}{x_i^\circ} < \tau \Rightarrow \quad s_i^\circ \rightarrow 0 \text{ and } x_i^\circ > 0$$

where  $\tau \in [0, 1]$  is a threshold. Note that if  $(x^\circ, s^\circ)$  is a perfectly-centered solution, hence for all  $i$ ,  $x_i^\circ s_i^\circ = \mu$ , then we can use the variables directly as an indicator since

$$\begin{aligned}
\frac{x_i^\circ}{s_i^\circ} < \tau &\Leftrightarrow \quad x_i^\circ < \sqrt{\mu\tau} < \sqrt{\mu/\tau} < s_i^\circ \\
\frac{s_i^\circ}{x_i^\circ} < \tau &\Leftrightarrow \quad s_i^\circ < \sqrt{\mu\tau} < \sqrt{\mu/\tau} < x_i^\circ.
\end{aligned}$$

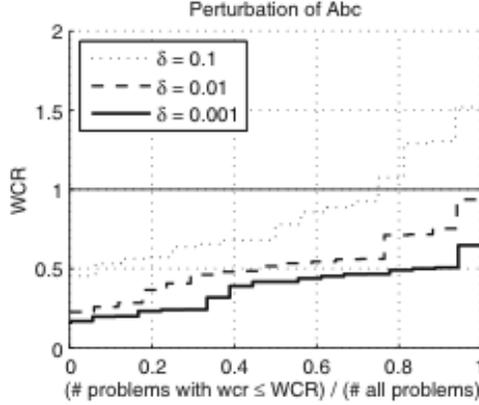
This allows us to consider only variables for relaxation, which values exceed  $\sqrt{\mu/\tau}$ . Extending (16), we initialize the residuals with

$$(19) \quad (r_{xi}^0, r_{si}^0) = \begin{cases} ((2\sqrt{\mu}\|(r_b, r_c, 1)\|_\infty + \|(r_b, r_c, 1)\|_\infty^2)/s_i^\circ, 0) & \text{if } x_i^\circ < \sqrt{\mu\tau} < \sqrt{\mu/\tau} < s_i^\circ \\ (0, (2\sqrt{\mu}\|(r_b, r_c, 1)\|_\infty + \|(r_b, r_c, 1)\|_\infty^2)/x_i^\circ) & \text{if } s_i^\circ < \sqrt{\mu\tau} < \sqrt{\mu/\tau} < x_i^\circ \\ (\sqrt{x_i^\circ/s_i^\circ}\|(r_b, r_c, 1)\|_\infty, \sqrt{s_i^\circ/x_i^\circ}\|(r_b, r_c, 1)\|_\infty) & \text{if } \sqrt{\mu\tau} \leq (x_i^\circ, s_i^\circ) \leq \sqrt{\mu/\tau}. \end{cases}$$

## 5. COMPUTATIONAL RESULTS

The above warmstarting scheme was tested by Engau *et. al.* on selected LP problems from the Netlib test suite [4]. For each test case, they solved the initial problem with data  $d^\circ$  for an optimal solution  $(x^\circ, y^\circ, s^\circ)$ . The following perturbation is described here for  $\Delta c$  and applies similarly to  $\Delta A$  and  $\Delta b$ . The entries in  $\Delta c$  are chosen such that at most 10% or 20 entries are changed. They then generate a uniformly distributed random number  $\varepsilon \in [-1, 1]$

Figure 1: WCR Performance Profile With Primal-Dual Indicator Threshold  $\tau = 1$ .



and set

$$\Delta c_i = \begin{cases} \varepsilon \delta & \text{if } c_i^\circ = 0 \\ \varepsilon \delta c_i^\circ & \text{otherwise} \end{cases}$$

where  $\delta$  is the perturbation level (scale factor). The scale factor balances the perturbations among problem sets that might be ill conditioned. The new perturbed problem is then solved by a cold start and a warmstart. Engau *et. al.* then computed the Warm-to-Cold-start-iteration-ratio (WCR). An example of the results is shown in Figure 5. From the results, Engau *et. al.* conclude that the warmstart scheme reduces the overall iterations of a cold start by over 50%.

## 6. CONCLUSION

In this report we summarized and covered parts of [4, Section 2], [4, Section 3], and [4, Section 4]. We did not cover [4, Section 1], which is a historical introduction to LPs and IPMs. We also omitted the part about shifted-barrier methods in [4, Section 2.2] as well as the complexity analysis in [4, Section 3.1].

Contrary to the popular belief, the research presented in the paper shows that IPMs can be warmstarted efficiently. This opens the door for a whole new area of research. In [3], they extend their approach to combinatorial optimization, specifically for the use in *Mixed Integer Linear Programs* (MILP).

It would be interesting to see how the approach of Engau *et. al.* compares to the approach in [1] in numerical tests. A further direction of research would be the comparison of the IPM warmstart approach with warmstarting the simplex method. There are many problem instances where IPMs outperform the simplex method. However, since the simplex method is very suitable for warmstarts, it would be interesting to see if and how long (how many warmstarts) it takes for the simplex to outperform the IPM method. It should also be compared to algorithms that cross-over from IPM to the simplex method in order to do the warmstarts.

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