PROJECTION METHODS WITH SUMMABLE PERTURBATIONS

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Introduction

We studied the paper of Dan Butnariu et. al. about the convergence of projection methods under summmable perturbations [2]. This report provides the background material and underlying theory to understand the proof presented in the paper.

This report is divided up into three principal sections. In Section 1 we introduce the theory of nonexpansive operators and attracting operators in general. In Section 2 we apply the results from Section 1 to the projection operator and to compositions of projections in order to obtain some important properties. In the last section, we use the properties of compositions of projections to proof the main theorem in [2].

To keep this report at a reasonable length, we will not discuss the applications and examples of the proof from the paper. We also assume that the reader has some basic knowledge about functional analysis and convexity.

It should be noted that the original proofs for [5, Theorem 1] and [3, Theorem 4] are in Banach space. However, for simplicity and due to the finite nature of the application, we restricted the setting of this report to the Euclidean space $\mathbb{R}^n$. For the remainder of the report, we use $\|\cdot\|$ to denote the Euclidean norm.

1. Attracting operators in $\mathbb{R}^n$.

Definition 1.1. Nonexpansive operators. Let C be a closed convex and nonempty subset of $\mathbb{R}^n$. A mapping $T : C \to \mathbb{R}^n$ is called *nonexpansive* if $\forall x, y \in C$

$$\|Tx - Ty\| \leq \|x - y\|.$$

Lemma 1.2. $\|\cdot\|^2$ *is strongly convex. That says*

$$\|\lambda a + (1-\lambda)b\|^2 = \lambda\|a\|^2 + (1-\lambda)\|b\|^2 - \lambda(1-\lambda)\|a-b\|^2$$

and if $a \neq b$ *we have*

$$\|\lambda a + (1-\lambda)b\|^2 < \lambda\|a\|^2 + (1-\lambda)\|b\|^2.$$

Proof. Let $a, b \in \mathbb{R}^n$ such that $a \neq b$ and $\lambda \in [0,1]$. Then

$$\begin{aligned}
\|\lambda a + (1-\lambda)b\|^2 &= \lambda^2\|a\|^2 + 2\lambda(1-\lambda)\langle a, b\rangle + (1-\lambda)^2\|b\|^2 \\
\Leftrightarrow \|\lambda a + (1-\lambda)b\|^2 - \lambda\|a\|^2 - (1-\lambda)\|b\|^2 &= (\lambda^2 - \lambda)\|a\|^2 + 2\lambda(1-\lambda)\langle a, b\rangle + ((1-\lambda)^2 \\
&\quad - (1-\lambda))\|b\|^2 \\
&= -\lambda(1-\lambda)\|a\|^2 + 2\lambda(1-\lambda)\langle a, b\rangle \\
&\quad + (1-\lambda)(-\lambda)\|b\|^2 \\
&= -\lambda(1-\lambda)(\|a\|^2 - 2\langle a, b\rangle + \|b\|^2) \\
&= -\lambda(1-\lambda)\|a-b\|^2 \\
\Leftrightarrow \|\lambda a + (1-\lambda)b\|^2 &= \lambda\|a\|^2 + (1-\lambda)\|b\|^2 - \lambda(1-\lambda)\|a-b\|^2.
\end{aligned}$$

As $a \neq b$, we have $\lambda(1-\lambda)\|a-b\|^2 > 0$ and

$$\|\lambda a + (1-\lambda)b\|^2 < \lambda\|a\|^2 + (1-\lambda)\|b\|^2.$$

□

Theorem 1.3. *A nonexpansive operator is continuous in $\mathbb{R}^n$.*

Proof. Let $(x_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$ and let $x_k \to x$ when $k \to \infty$. Then, since T is nonexpansive,

$$0 \leq \|Tx_k - Tx\| \leq \|x_k - x\| \to 0.$$

Hence by the squeeze Theorem, $Tx_k \to Tx$ as $x_k \to x$ and T is continuous. □

Theorem 1.4. *If T is a nonexpansive mapping as defined in Definition 1.1, then the set of all* fixed points *Fix T defined by*

$$\operatorname{Fix} T = \{x \in C | x = Tx\}$$

is always closed and convex.

Proof. Let $x, y \in \operatorname{Fix} T, \lambda \in [0,1]$ and $f = \lambda x + (1-\lambda)y$. Then $f \in C$ since C is convex. Now by Lemma 1.2 and the nonexpansivity of T we have

$$\begin{aligned}
\|Tf - f\|^2 &= \|Tf - \lambda x - (1-\lambda)y\|^2 \\
&= \|\lambda Tf + (1-\lambda)Tf - \lambda Tx - (1-\lambda)Ty\|^2 \\
&= \|\lambda(Tf - Tx) + (1-\lambda)(Tf - Ty)\|^2 \\
&= \lambda\|Tf - Tx\|^2 + (1-\lambda)\|Tf - Ty\|^2 - \lambda(1-\lambda)\|Tx - Ty\|^2 \\
&\leq \lambda\|f - x\|^2 + (1-\lambda)\|f - y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \\
&= \|\lambda(f - x) + (1-\lambda)(f - y)\|^2 = \|f - \lambda x - (1-\lambda)y\|^2 \\
&= \|f - f\|^2 = 0.
\end{aligned}$$

Hence $\operatorname{Fix} T$ is convex. Now let $(x_k)_{k=1}^{\infty} \in \operatorname{Fix} T$ and $x_k \to x \in \mathbb{R}^n$. Since $\operatorname{Fix} T \subseteq C$ and C is closed, $x \in C$. By Theorem 1.3 T is continuous and $Tx_k \to Tx$ as $x_k \to x$. Since $(x_k)_{k=1}^{\infty} \in \operatorname{Fix} T$ we have $Tx_k \to x$. So $Tx = x$ and it follows that $x \in \operatorname{Fix} T$. Hence $\operatorname{Fix} T$ is closed. □

Definition 1.5. Attracting operators. Let C and $\operatorname{Fix} T$ be closed convex and nonempty subsets of $\mathbb{R}^n$ and $\operatorname{Fix} T \subseteq C$. A mapping $T : C \to C$ is called *attracting with respect to* $\operatorname{Fix} T$ if $\forall x \in C \setminus \operatorname{Fix} T$ and $f \in \operatorname{Fix} T$

$$\|Tx - f\| < \|x - f\|.$$

Definition 1.6. Strongly attracting operators. Let C and $\operatorname{Fix} T$ be closed convex and nonempty subsets of $\mathbb{R}^n$ and $\operatorname{Fix} T \subseteq C$. A mapping $T : C \to C$ is called *strongly attracting* or *κ-attracting* with respect to $\operatorname{Fix} T$ if there is some $\kappa > 0$ such that $\forall x \in C \setminus \operatorname{Fix} T$ and $f \in \operatorname{Fix} T$

$$\kappa\|Tx - x\|^2 \leq \|x - f\|^2 - \|Tx - f\|^2.$$

We can see that strongly or κ-attracting operators are a subset of attracting operators. That is

$$\begin{aligned} \kappa\|Tx - x\|^2 &\leq \|x - f\|^2 - \|Tx - f\|^2 \\ \Rightarrow \kappa\|Tx - x\|^2 + \|Tx - f\|^2 &\leq \|x - f\|^2 \\ \Rightarrow \|Tx - f\|^2 &< \|x - f\|^2 \qquad \text{if } x \neq Tx \end{aligned}$$

and taking the square root on each side makes it attracting.

Definition 1.7. Firmly nonexpansive operators. Let C be closed convex and nonempty subsets of $\mathbb{R}^n$. A mapping $T : C \to C$ is called *firmly nonexpansive* if $\forall x, y \in C$

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle.$$

Note that to show that the set of firmly nonexpansive mappings is a subset of nonexpansive mappings, one can use the Cauchy-Schwarz inequality ($\langle a, b\rangle \leq \|a\|\|b\|$):

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \langle Tx - Ty, x - y\rangle \\ \Rightarrow \|Tx - Ty\|^2 &\leq \|Tx - Ty\|\|x - y\| \end{aligned}$$

and divide both sides with $\|Tx - Ty\|$

$$\|Tx - Ty\| \leq \|x - y\|.$$

In fact, the set of firmly nonexpansive operators is a subset of κ-attracting operators. The proof can be found in [1]. We will show a special version of that proof in the next section.

Theorem 1.8 (Bauschke and Borwein [1, Proposition 2.10 (i)])**.** *Let C be a closed convex and nonempty subset of $\mathbb{R}^n$ and $T_1, \ldots, T_N : C \to C$ are attracting mappings. If $\cap_{i=1}^N \operatorname{Fix} T_i \neq \varnothing$ then*

$$\operatorname{Fix}(T_N \circ T_{N-1} \circ \cdots \circ T_1) = \cap_{i=1}^N \operatorname{Fix} T_i$$

and $T_N \circ T_{N-1} \circ \cdots \circ T_1$ is an attracting mapping.

Proof. Without loss of generality, we assume $N = 2$. The result for $N > 2$ follows by induction.

Assume $f \in \operatorname{Fix} T_1 \cap \operatorname{Fix} T_2$. Then $T_2(T_1 f) = T_2 f = f$. So $f \in \operatorname{Fix}(T_2 \circ T_1)$ and $\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2 \subseteq \operatorname{Fix}(T_2 \circ T_1)$.

We now show the other inclusion. Let $f \in \operatorname{Fix}(T_2 \circ T_1)$. Hence $f = T_2(T_1 f)$. We claim that $f \in \operatorname{Fix} T_1$. Suppose not. We let $f' \in \operatorname{Fix} T_1 \cap \operatorname{Fix} T_2$. Since T_2 is attracting, by Definition 1.5

we have

$$\|f - f'\| = \|T_2(T_1 f) - f'\| \le \|T_1 f - f'\| < \|f - f'\|.$$

A contradiction. So $f \in \operatorname{Fix} T_1$ and $T_2(T_1 f) = T_2 f = f$. Hence f is also in $\operatorname{Fix} T_2$ and $\operatorname{Fix}(T_2 \circ T_1) \subseteq \operatorname{Fix} T_1 \cap \operatorname{Fix} T_2$. Altogether

$$\operatorname{Fix}(T_2 \circ T_1) = \operatorname{Fix} T_1 \cap \operatorname{Fix} T_2.$$

It remains to show that $T_2 \circ T_1$ is attracting. Let $f \in \operatorname{Fix}(T_2 \circ T_1)$ and $x \in C$ but $x \notin \operatorname{Fix}(T_2 \circ T_1)$. We have two cases.

$T_1 x = x$.: Since x is not a fixed point of $T_2 \circ T_1$, $T_2 x \neq x$ and hence

$$\|T_2(T_1 x) - f\| = \|T_2 x - f\| < \|x - f\|.$$

$T_1 x \neq x$.: Hence x is not a fixed point of either operator and

$$\|T_2(T_1 x) - f\| \le \|T_1 x - f\| < \|x - f\|.$$

In both cases, $T_2 \circ T_1$ is attracting. □

Theorem 1.9 (Bauschke and Borwein [1, Proposition 2.10 (ii)]). *Let $T_1, \ldots, T_N : C \to C$ be defined as in Theorem 1.8. If every $T_i, i = 1, \ldots, N$ is κ_i-attracting, then $T_N \circ T_{N-1} \circ \cdots \circ T_1$ is κ-attracting with*

$$\kappa = \frac{\min\{\kappa_1, \ldots, \kappa_N\}}{2^{N-1}}.$$

Proof. As in Theorem 1.8, we use $N = 2$ and let the reader proof the case $N > 2$ by induction.

Let $x \in C, f \in$ Fix $(T_2 \circ T_1)$. Using the parallelogram law[1], we have

$$\begin{aligned}
\|x - T_2(T_1 x)\|^2 &= \|x - T_1 x + T_1 x - T_2(T_1 x)\|^2 \\
&\le (\|x - T_1 x\| + \|T_1 x - T_2(T_1 x)\|)^2 \\
&\le 2(\|x - T_1 x\|^2 + \|T_1 x - T_2(T_1 x)\|^2) \\
&\le \frac{2}{\kappa_1}(\|x - f\|^2 - \|T_1 x - f\|^2) + \frac{2}{\kappa_2}(\|T_1 x - f\|^2 - \|T_2(T_1 x) - f\|^2) \\
&\le \frac{2}{\min\{\kappa_1, \kappa_2\}}(\|x - f\|^2 - \|T_2(T_1 x) - f\|^2).
\end{aligned}$$

□

[1]Parallelogram law:

$$\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2 \\
\Leftrightarrow (a+b)^2 + a^2 + b^2 &= 2a^2 + 2b^2 + 2ab \\
\Leftrightarrow (a+b)^2 &= 2a^2 + 2b^2 - a^2 + 2ab - b^2 \\
&= 2a^2 + 2b^2 - (a-b)^2.
\end{aligned}$$

Since $(a-b)^2 \ge 0$, we have

$$(a+b)^2 \le 2a^2 + 2b^2$$

2. Projections and compositions of projections in $\mathbb{R}^n$

Definition 2.1. Let C be a closed and convex subset of $\mathbb{R}^n$ and let the norm be strictly convex (*i.e*, the Euclidean norm). The operator $P_C : \mathbb{R}^n \to C$ is called a *projection from* $\mathbb{R}^n$ *onto* C, if $\forall x \in \mathbb{R}^n, P_C(x) \in C$

$$\|x - P_C(x)\| = \inf\{\|x - y\| | y \in C\}.$$

A proof of existence and uniqueness of $P_C(x)$ can be found in [6, Theorem 2.10]. We show a similar proof in Theorem 3.9.

We now classify projections and compositions of projections according to the definitions shown in Section 1.

Lemma 2.2. *If C is a convex, closed, and nonempty subset of $\mathbb{R}^n$, and $P_C(x)$ is a projection of $x \in \mathbb{R}^n$ onto C with $P_C(x) \in C$, then $\forall x \in \mathbb{R}^n, \forall f \in C$,*

$$\langle f - P_C(x), x - P_C(x)\rangle \leq 0.$$

Proof. By definition, if $P_C(x)$ is the projection for some $x \in \mathbb{R}^n$ onto C, then $\forall f \in C$ we have

$$\|P_C(x) - x\|^2 \leq \|f - x\|^2.$$

Consider $\theta \in [0, 1]$. Thus $\theta f + (1 - \theta)P_C(x) \in C$. Hence

$$\begin{aligned}
\|P_C(x) - x\|^2 &\leq \|\theta f + (1 - \theta)P_C(x) - x\|^2 \\
&= \|P_C(x) + \theta(f - P_C(x)) - x\|^2 \\
&= \langle P_C(x) + \theta(f - P_C(x)) - x, P_C(x) + \theta(f - P_C(x)) - x\rangle \\
&= \|P_C(x) - x\|^2 - 2\theta\langle f - P_C(x), x - P_C(x)\rangle + \theta^2\|f - P_C(x)\|^2.
\end{aligned}$$

Subtracting $\|P_C(x) - x\|^2$ on both sides and rearranging the terms yields

$$2\theta\langle f - P_C(x), x - P_C(x)\rangle \leq \theta^2\|f - P_C(x)\|^2.$$

We divide each side by 2θ and let $\theta \to 0$. Hence

$$\langle f - P_C(x), x - P_C(x)\rangle \leq 0.$$ □

Theorem 2.3. *If C is a convex, closed, and nonempty subset of $\mathbb{R}^n$ and $P_C(x)$ is a projection of x onto C, then $P_C(x)$ is firmly nonexpansive. That is for any $x, y \in \mathbb{R}^n$,*

$$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle.$$

Proof. Let $x, y \in \mathbb{R}^n$ and $f \in C$. According to Lemma 2.2 we can write

$$\langle f - P_C(x), x - P_C(x)\rangle \leq 0 \tag{1}$$

$$\langle f - P_C(y), y - P_C(y)\rangle \leq 0 \tag{2}$$

Now we plug $f = P_C(y)$ into (1) and $f = P_C(x)$ into (2) and add the two equations together and we get

$$\langle P_C(x) - P_C(y), P_C(x) - P_C(y)\rangle + \langle P_C(y) - P_C(x), x - y\rangle \leq 0$$

and hence

$$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle. \qquad \square$$

As mentioned before, a firmly nonexpansive operator is also κ-attracting. We will show this in the next Theorem.

Theorem 2.4. *The projection P_C is 1-attracting with respect to C. That is $\forall x \in \mathbb{R}^n, \forall f \in C$,*

$$\|f - P_C(x)\|^2 + \|P_C(x) - x\|^2 \leq \|f - x\|^2.$$

Proof.

$$\begin{aligned}\|f - x\|^2 &= \|f - P_C(x) + P_C(x) - x\|^2 \\ &= \langle f - P_C(x) - x + P_C(x), f - P_C(x) - x + P_C(x)\rangle \\ &= \|f - P_C(x)\|^2 - 2\langle f - P_C(x), x - P_C(x)\rangle + \|x - P_C(x)\|^2\end{aligned}$$

By Lemma 2.2, we have $2\langle f - P_C(x), x - P_C(x)\rangle \leq 0$. Hence

$$\|f - P_C(x)\|^2 + \|x - P_C(x)\|^2 \leq \|f - P_C(x)\|^2 - 2\langle f - P_C(x), x - P_C(x)\rangle + \|x - P_C(x)\|^2$$

and therefore

$$\|f - P_C(x)\|^2 + \|x - P_C(x)\|^2 \leq \|f - x\|^2. \qquad \square$$

Corollary 2.5. *Since P_C is 1-attracting, by Theorem 1.9, A composition of projection operators is κ-attracting. In particular, if $\mathbf{P} = P_{C_N} \circ P_{C_{N-1}} \circ \cdots P_{C_1}$ then $\mathbf{P}$ is κ-attracting with*

$$\kappa = \frac{\min\{\kappa_1, \ldots, \kappa_N\}}{2^{N-1}} = \frac{1}{2^{N-1}} = 2^{-(N-1)}.$$

3. Convergence of projection methods with and without summable perturbations in $\mathbb{R}^n$

With the previous sections, we now have enough background to understand the proof in [2]. Let $C_i \subseteq \mathbb{R}^n$ for $i = 1, \ldots, m$ be closed and convex sets. We want to find a point in

$$C = \cap_{i=1}^m C_i.$$

We start by defining the *amalgamated projection method* and then show the proof of convergence of the amalgamtated projection method to C followed by a proof of convergence to C if the projection method is subject to summable perturbations.

Definition 3.1. We call the tuple (Ω, ω) an *amalgamator*, where Ω is a finite set of index vectors. Each index vector $t = (t_1, \ldots, t_p)$ with $t_i \in \{C_1, \ldots, C_m\}$ for $i \in \{1, \ldots, p\}$ represents a composition of projections of the form

$$P[t] = P_{t_p} \circ \ldots \circ P_{t_1}.$$

The weight function $\omega : \Omega \to \mathbb{R}_{++}$ assigns a weight $\omega(t)$ to each composition and satisfies

$$\sum_{t \in \Omega} \omega(t) = 1.$$

Definition 3.2. If (Ω, ω) is an amalgamator as in Definition 3.1, we define the *amalgamated projection operator* $\mathbf{P} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\mathbf{P}x = \sum_{t \in \Omega} \omega(t) P[t]x.$$

The *amalgamated projection method* is defined as

$$x_0 \in \mathbb{R}^n$$
$$x_{k+1} = \mathbf{P}x_k, \quad \forall k \in \mathbb{N}.$$

To proof the convergence of the amalgamated projection operator $\mathbf{P}$ to the set C, $\mathbf{P}$ needs to have three essential properties. We need to show that the amalgamated projection operator is nonexpansive, that the fixed point set $\operatorname{Fix} \mathbf{P} = C$, and that $\mathbf{P}$ is *asymptotically regular* (as defined later).

From Corollary 2.5 we know that every composition of projections $P[t]$ is κ-attracting and therefore nonexpansive. We now show that the amalgamated projection operator is nonexpansive and that the fixed point set $\operatorname{Fix} \mathbf{P} = C$.

Lemma 3.3. *The amalgamated projection operator $\mathbf{P}$ as in Definition 3.2 is nonexpansive.*

Proof. We have $\mathbf{P}x = \sum_{t \in \Omega} \omega(t) P[t]x$ with $P[t]$ nonexpansive. Hence for any $P[t]$ we have

$$\|P[t]x - P[t]y\| \leq \|x - y\|.$$

It follows that for any scalar $\omega(t)$ we have

$$\|\omega(t)P[t]x - \omega(t)P[t]y\| \leq \|\omega(t)x - \omega(t)y\|$$

Hence

$$\|\sum_{t \in \Omega} \omega(t)P[t]x - \sum_{t \in \Omega} \omega(t)P[t]y\| \leq \|\sum_{t \in \Omega} \omega(t)x - \sum_{t \in \Omega} \omega(t)y\|$$

and therefore

$$\|\mathbf{P}x - \mathbf{P}y\| \leq \|x - y\|.$$ □

Lemma 3.4. *Let $C = \cap_{i=1}^m C_i$ be closed, convex, and not empty, If the set of index vectors Ω in an amalgamator contains for every $C_i, i \in \{1, \ldots, m\}$ at least one index vector that has C_i as an element, then*

$$\operatorname{Fix} \mathbf{P} = C.$$

Proof.

$$C \subseteq \operatorname{Fix} \mathbf{P} = \bigcap_{t \in \Omega} \operatorname{Fix} P[t] = \bigcap_{i=1}^{m} \bigcap_{t \in \Omega_i} \operatorname{Fix} P[t] \subseteq \bigcap_{i=1}^{m} C_i = C.$$ □

We now show that if a composition of projection operators is *asymptotically regular*, it will converge to a fixpoint. The proof will be based on Opial's general version for Banach spaces. We first define *asymptotically regular*.

Definition 3.5. A sequence $(x_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$ is *asymptotically regular* if

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0.$$

We now show that the amalgamated projection operator $\mathbf{P}$ is asymptotically regular.

Lemma 3.6. *The amalgamated projection operator* $\mathbf{P}$ *as in Definition 3.2 is asymptotically regular.*

Proof. Let $i \in \{1, \dots, m\}$. P_{C_i} is 1-attracting with respect to C_i. That is

$$\|c - P_{C_i}(x)\|^2 + \|P_{C_i}(x) - x\|^2 \leq \|c - x\|^2$$

$\forall c \in C_i$ and $x \in \mathbb{R}^n$.
Hence for $P[t]$, if $p(t)$ is the length of the index vector t,

$$\kappa = \frac{\min\{\kappa_{t_1}, \dots, \kappa_{t_{p(t)}}\}}{2^{p(t)-1}} = \frac{1}{2^{p(t)-1}} = 2^{-p(t)+1}$$

and $P[t]$ is $2^{-p(t)+1}$-attracting with respect to C. That is $\forall c \in C$ and $x \in \mathbb{R}^n$

$$\|c - P[t]x\|^2 + 2^{-p(t)+1}\|P[t]x - x\|^2 \leq \|c - x\|^2.$$

In particular, $\forall c \in C$ and $k \in \mathbb{N}$, we have

$$\|c - P[t]x_k\|^2 + 2^{-p(t)+1}\|P[t]x_k - x_k\|^2 \leq \|c - x_k\|^2.$$

If we let $\alpha = 2^{-\max_{t\in\Omega} p(t) - 1}$, then

$$\|c - P[t]x_k\|^2 + \alpha\|P[t]x_k - x_k\|^2 \leq \|c - x_k\|^2.$$

Hence $\forall c \in C$ and $k \in \mathbb{N}$,

$$\sum_{t\in\Omega} \omega(t) \cdot \left(\|c - P[t]x_k\|^2 + \alpha\|P[t]x_k - x^k\|^2\right) \leq \sum_{t\in\Omega} \omega(t) \cdot \|c - x_k\|^2.$$

Now $f(x) = \|u - x\|^2$ is convex, so $f(\sum \lambda_i x_i) \leq \sum \lambda_i f(x_i)$. Since $\sum \omega(t) = 1$ we have

$$\|c - \sum \omega(t)P[t]x_k\|^2 + \alpha\|\sum \omega(t)P[t]x_k - x_k\|^2 \leq \|c - x_k\|^2.$$

Using $\mathbf{P} = \sum \omega(t)P[t]x_k$

$$\|c - \mathbf{P}x_k\|^2 + \alpha\|\mathbf{P}x_k - x_k\|^2 \leq \|c - x_k\|^2.$$

and with $x_{k+1} = \mathbf{P}x_k$

$$\|c - x_{k+1}\|^2 + \alpha\|x_{k+1} - x_k\|^2 \leq \|c - x_k\|^2. \tag{3}$$

As $\|\cdot\|^2$ and α are nonnegative,

$$\|c - x_{k+1}\| \leq \|c - x_k\|$$

and $(\|c - x_k\|)_{k=0}^{\infty}$ converges. If we rearrange equation (3), we have

$$\alpha\|x_{k+1} - x_k\|^2 \leq \|c - x_k\|^2 - \|c - x_{k+1}\|^2 \to 0.$$

It follows that

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0.$$

□

We can now use Opial's proof of convergence for nonexpansive and asymptotically regular operators. In the next step, we establish the proofs of two Lemmas that we need in Opial's proof.

Lemma 3.7. *If a sequence $(x_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$ converges to x^*, then for $x \neq x^*$ we have*

$$\lim_{k\to\infty} \|x_k - x\| > \lim_{k\to\infty} \|x_k - x^*\|$$

Proof. The norm $\|\cdot\|$ is continuous. We therefore have

$$\lim_{k\to\infty} \|x_k - x\| = \|\lim_{k\to\infty} x_k - x\| = \|x^* - x\| > 0 = \|x^* - x^*\| = \|\lim_{k\to\infty} x_k - x^*\| = \lim_{k\to\infty} \|x_k - x^*\|.$$

□

Lemma 3.8. *Let C be a closed convex subset of $\mathbb{R}^n$. If a mapping $T : C \to \mathbb{R}^n$ is nonexpansive, then for any sequence $(x_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$, $x_k \to x^*$ and $(x_k - Tx_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$, $x_k - Tx_k \to y^*$, we have*

$$y^* = x^* - Tx^*$$

Proof. By Theorem 1.3, T is continuous. Hence as $x_k \to x^*$ and $x_k - Tx_k \to y^*$ we conclude that $x^* - Tx^* = y^*$. □

Theorem 3.9 (Opial [5, Theorem 1]). *Let C be a closed convex set in $\mathbb{R}^n$ and $T : C \to C$ a nonexpansive asymptotically regular mapping for which $\operatorname{Fix} T \neq \varnothing$. Then, $\forall x \in \mathbb{R}^n$, $T^k x \to y$ when $k \to \infty$ where $y \in \operatorname{Fix} T$.*

Proof. $\forall y \in \operatorname{Fix} T$. the sequence $(\|T^k x - y\|)$ is nonincreasing, since by the nonexpansivity of T we have

$$\|T^{k+1}x - y\| = \|T(T^k x) - Ty\| \le \|T^k x - y\| \qquad (k = 0, 1, \ldots).$$

Therefore, $\|T^k x - y\|$ is nonincreasing and $(T^k x)$ is bounded. Since $\|T^k x - y\|$ is nonincreasing and T is continuous (by Theorem 1.3), it follows that for any $y \in \operatorname{Fix} T$, there exists the nonnegative limit

$$d(y) = \lim_{k\to\infty} \|T^k x - y\|^2.$$

Now consider $y_1, y_2 \in C, y_1 \neq y_2$, and $\lambda \in (0, 1)$. With the use of the identity in Lemma 1.2 we have

$$\begin{aligned}\|\lambda y_1 + (1-\lambda)y_2 - T^k x\|^2 &= \|\lambda(y_1 - T^k x) + (1-\lambda)(y_2 - T^k x)\|^2 \\ &= \lambda\|y_1 - T^k x\|^2 + (1-\lambda)\|y_2 - T^k x\|^2 - \lambda(1-\lambda)\|y_1 - y_2\|^2\end{aligned}$$

We take the limit on each side and obtain

$$\begin{aligned}\lim_{k\to\infty} \|\lambda y_1 + (1-\lambda)y_2 - T^k x\|^2 = \lim_{k\to\infty} \lambda\|y_1 - T^k x\|^2 + \lim_{k\to\infty} (1-\lambda)\|y_2 - T^k x\|^2 \\ - \lambda(1-\lambda)\|y_1 - y_2\|^2.\end{aligned}$$

So

$$d(\lambda y_1 + (1-\lambda)y_2) = \lambda d(y_1) + (1-\lambda)d(y_2) - \lambda(1-\lambda)\|y_1 - y_2\|^2$$

and since $y_1 \neq y_2$ we have $\lambda(1-\lambda)\|y_1 - y_2\|^2 > 0$ and hence

$$d(\lambda y_1 + (1-\lambda)y_2) < \lambda d(y_1) + (1-\lambda)d(y_2).$$

So $d(\cdot)$ is a strictly convex function. It follows that $d(\cdot)$ has a unique minimizer. Let y be that unique minimizer such that $\delta = d(y)$.

Now we show that $T^k x \to y$ when $k \to \infty$. Suppose the contrary. Then, since $(T^k x)$ is bounded, by the Bolzano-Weierstrass theorem, there exists a subsequence $(T^{k_j} x)$ where $T^{k_j} x \to z$ when $k_j \to \infty$. By our assumption, $z \neq y$. Since T is asymptotically regular, we have

$$\lim_{k_j \to \infty} \|T^{k_j+1} x - T^{k_j} x\| = 0$$

and by Lemma 3.8 we have

$$Tz - z = 0$$

and $z \in \operatorname{Fix} T$. Now, by Lemma 3.7, we have

$$\delta = d(y) = \lim_{k_j \to \infty} \|T^{k_j} x - y\| > \lim_{k_j \to \infty} \|T^{k_j} x - z\| = d(z).$$

This contradicts the claim that y is the unique minimizer and δ the smallest distance between any point in $\operatorname{Fix} T$ and $T^k x$ when $k \to \infty$. Therefore $T^k x \to y$ when $k \to \infty$. □

We showed in Lemma 3.3 that the amalgamated projection operator $\mathbf{P}$ is nonexpansive. In Lemma 3.4 we showed that $\operatorname{Fix} \mathbf{P} = C$ and in Lemma 3.6 we established the fact that $\mathbf{P}$ is asymptotically regular. We can therefore conclude that Opial's Theorem (Theorem 3.9) applies to $\mathbf{P}$ and $\mathbf{P}$ converges to C.

The next part is the main result in [2]. We first state the main theorem and we then give the proof that was shown in an earlier paper [3] together with the steps that are shown in [2].

Theorem 3.10 (Butnariu et. al. [3, Theorem 4]). *Let C_i, $1 \leq i \leq m$, be closed convex sets in $\mathbb{R}^n$ with a nonempty intersection C. If $(\beta_k)_{k=0}^{\infty}$ is a sequence of positive real numbers such that $\sum_{k=0}^{\infty} \beta_k < \infty$ and $(v_k)_{k=0}^{\infty}$ is a bounded sequence of vectors in $\mathbb{R}^n$, then for any amalgamator (Ω, ω) the procedure*

$$\begin{aligned} &x_0 \in \mathbb{R}^n \\ &x_{k+1} = \mathbf{P}(x_k + \beta_k v_k), \qquad \forall k \in \mathbb{N} \end{aligned} \tag{4}$$

converges, and its limit $y^* \in \operatorname{Fix} \mathbf{P}$ is in C.

Proof. We let $(x_k)_{k=1}^{\infty}$ bet the sequence such that $x_{k+1} = \mathbf{P}(x_k + \beta_k v_k)$. We call this sequence the *inexact orbit* for $\mathbf{P}$. (An *exact orbit* of $\mathbf{P}$ has $\beta_k = 0$). We have

$$\begin{aligned} \|x_{k+1} - \mathbf{P}x_k\| = \|\mathbf{P}(x_k + \beta_k v_k) - \mathbf{P}x_k\| &\leq \|(x_k + \beta_k v_k) - x_k\| \\ &= \beta_k \|v_k\|. \end{aligned}$$

Since (v_k) is bounded by some upper bound M,

$$\sum_{k=0}^{\infty} \|x_{k+1} - \mathbf{P}x_k\| \leq M \sum_{k=0}^{\infty} \beta_k < \infty,$$

and $(\|x_{k+1} - \mathbf{P}x_k\|)_{k=0}^{\infty}$ is summable. Hence there exists a sequence $(r_k)_{k=0}^{\infty} \in \mathbb{R}$ such that $\sum_{k=0}^{\infty} r_k < \infty$ and

$$\|x_{k+1} - \mathbf{P}x_k\| \leq r_k. \tag{5}$$

We now show by induction that for all $i \in \mathbb{Z}_+$,

$$\|\mathbf{P}^i x_k - x_{k+i}\| \le (\sum_{j=k-1}^{i+k-1} r_j) - r_{k-1}. \tag{6}$$

Clearly, this holds for $i = 0$. Now we consider the case $i+1$. By (5) we have

$$\begin{aligned}\|x_{k+i+1} - \mathbf{P}^{i+1}x_k\| &\le \|x_{k+i+1} - \mathbf{P}x_{k+i}\| + \|\mathbf{P}x_{k+i} - \mathbf{P}^{i+1}x_k\| \\ &\le r_{k+i} + \|\mathbf{P}x_{k+i} - \mathbf{P}(\mathbf{P}^i x_k)\|.\end{aligned}$$

Since $\mathbf{P}$ is nonexpansive and by our assumption (6), we have

$$\begin{aligned}\|x_{k+i+1} - \mathbf{P}^{i+1}x_k\| &\le r_{k+i} + \|x_{k+i} - \mathbf{P}^i x_k\| \\ &\le r_{k+i} + \sum_{j=k-1}^{i+k-1} r_j - r_{k-1} = (\sum_{j=k-1}^{i+k} r_j) - r_{k-1}.\end{aligned}$$

So (6) holds for every integer $i \ge 0$. We now fix an integer $q \ge 1$. Since

$$\sum_{j=k-1}^{q+k} r_j - r_{k-1} = \sum_{j=k}^{k+q} r_j \le \sum_{j=k}^{\infty} r_j,$$

we have, using (6),

$$\|\mathbf{P}^q x_k - x_{k+q}\| \le \sum_{j=k}^{\infty} r_j. \tag{7}$$

Again, since $\mathbf{P}$ is nonexpansive we have for all integers $i \ge 1$

$$\|\mathbf{P}^{q+i}x_k - \mathbf{P}^i x_{k+q}\| = \|\mathbf{P}^i(\mathbf{P}^q x_k) - \mathbf{P}^i x_{k+q}\| \le \|\mathbf{P}^q x_k - x_{k+q}\| \le \sum_{j=k}^{\infty} r_j. \tag{8}$$

Now if we fix x_k, by Theorem 3.9, since $\mathbf{P}$ is asymptotically regular, $\mathbf{P}^i x_k \to y_k$ where $y_k \in \operatorname{Fix}\mathbf{P}$. So as $i \to \infty$, (8) becomes

$$\|y_k - y_{k+q}\| \le \sum_{j=k}^{\infty} r_j.$$

This holds for all integers q and k and since $\sum_{j=k}^{\infty} r_j < \infty$, we can see that $(y_k)_{k=1}^{\infty}$ is a Cauchy sequence.

Now $\mathbb{R}^n$ is a complete metric space [4, Ex. 1.5-1]. It follows that $y_q \to y^* \in \mathbb{R}^n$ as $q \to \infty$ [4, Definition 1.4-3]. So if $q \to \infty$ then for all $k \ge 1$

$$\|y_k - y^*\| \le \sum_{j=k}^{\infty} r_j \tag{9}$$

and $y_k \to y^*$ as $k \to \infty$.

We now show that the inexact orbit $(x_k)_{k=1}^{\infty}$ converges to y^*. Fix $\varepsilon > 0$. From (5) we know that there exists k_0 such that $\sum_{j=k}^{\infty} r_j < \varepsilon/3$ when $k > k_0$. Hence from (7) and (9) we have

$$\begin{aligned}\|y^* - x_{k+i}\| &\le \|y^* - y_k\| + \|y_k - \mathbf{P}^i x_k\| + \|\mathbf{P}^i x_k - x_{k+i}\| \\ &\le \sum_{j=k}^{\infty} r_j + \|y_k - \mathbf{P}^i x_k\| + \sum_{j=k}^{\infty} r_j < \frac{\varepsilon}{3} + \|y_k - \mathbf{P}^i x_k\| + \frac{\varepsilon}{3}\end{aligned}$$

for any integer $i \ge 1$. Now $\mathbf{P}^i x_k \to y_k$ as $i \to \infty$. Therefore there exists an integer i_0 such that

$$\|y_k - \mathbf{P}^i x_k\| < \frac{\varepsilon}{3}$$

when $i > i_0$. So for all $k > k_0, i > i_0$ we have

$$\|y^* - x_{k+i}\| < \varepsilon.$$

We conclude that $x_k \to y^*$ when $k \to \infty$. By Theorem 1.3, $\mathbf{P}$ is continuous and since $y_k \in \operatorname{Fix}\mathbf{P}$

$$y^* = \lim_{k\to\infty} y_k = \lim_{k\to\infty} \mathbf{P} y_k = \mathbf{P} y^*.$$

Therefore $y^* \in \operatorname{Fix}\mathbf{P}$. □

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